LECTURE 38 INDEFINITE INTEGRALS AND THE SUBSTITUTION METHOD

From the Fundamental Theorem of Calculus, we learned that given a continuous function $f(x)$ on $[a,b]$, the definite integral

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

where F is one antiderivative of f. From antidifferentiation, we learned that

$$\int f(x)\,dx = F(x) + C.$$

We are well-equipped to solve all kind of integrals.

Example.

$$\int \left(x^3+x\right)^5 \left(3x^2+1\right) dx.$$

Solution. Umm .... an indefinite integral. Let's find the antiderivative of

$$f(x) = \left(x^3+x\right)^5 \left(3x^2+1\right).$$

So, what function has a derivative that looks like this? It's actually a tricky business!

Now, looking at $f(x)$, we see it is a product of two functions $\left(x^3+x\right)^5$ and $\left(3x^2+1\right)$. So, what kind of differentiation rule would possibly lead to a product? The chain rule. Then, the antiderivative must be written as a function composition,

$$F(x) = g(h(x)) \implies F'(x) = f(x) = g'(h(x))\,h'(x).$$

We may pair these up with

$$g'(h(x)) = \left(x^3+x\right)^5$$

and

$$h'(x) = 3x^2+1.$$

From this, can we guess the function form of g'? Yes. By looking at $h'(x) = 3x^2+1$, we figure that $h(x) = x^3+x+C$ for some constant C. However, we want $g'(h(x)) = \left(x^3+x\right)^5$, which means $C=0$ and $g'(x) = x^5$. Now, $g'(x) = x^5 \implies g(x) = \frac{x^6}{6} + C$. Altogether,

$$F(x) = g(h(x)) = \frac{\left(x^3+x\right)^6}{6} + C.$$

Now, let's put more structure into how to identify an antiderivative when we see things like this. We start with the chain rule itself, for $u(x)$

$$\frac{d}{dx}\left(\frac{u^{n+1}}{n+1}\right) = u^n \frac{du}{dx},$$

which implies

$$\frac{u^{n+1}}{n+1} + C = \int u^n \frac{du}{dx} dx = \int u^n du$$

Therefore, it is crucial to identify what u is when $f(x)$ is presented in the above fashion. At the same time, we must check $du = \frac{du}{dx}dx = u'(x)\,dx$ and see if it is accompanied along. Here,

$$\int f(x)\,dx = \int \left(x^3+x\right)^5 \left(3x^2+1\right) dx$$

where we identified

$$u(x) = x^3+x, \quad du = u'(x)\,dx = \left(3x^2+1\right) dx$$

using knowledge of differentials. Thus,

$$\int \left(x^3+x\right)^5 \left(3x^2+1\right) dx = \int u^5 du = \frac{u^6}{6} + C = \frac{\left(x^3+x\right)^6}{6} + C.$$

Example. Find $\int \sqrt{2x+1}dx$.

Solution. Note first that this integral does not fit the formula

$$\int u^n du$$

with $u = (2x+1)$ and $n = \frac{1}{2}$ since

$$du = \frac{du}{dx}dx = 2dx$$

which is not exactly dx. However, when we don't have something, we create it and compensate it.

$$\begin{aligned} \int \sqrt{2x+1}dx &= \frac{1}{2}\int \sqrt{2x+1}2dx \\ &= \frac{1}{2}\int u^{\frac{1}{2}} du \\ &= \frac{1}{2}\left(\frac{2}{3}u^{\frac{3}{2}}\right) + C \\ &= \frac{1}{3}(2x+1)^{\frac{3}{2}} + C \end{aligned}$$

This method is called the substitution method, that is, we must find the substitution function $\underline{u(x)}$so that the original now looks simpler with u in it with du as the integrating differential (instead of dx). It is a dual care process where you find a suitable u such that $du = u'(x)\,dx$ also shows up in the original integral.

Theorem. *If $u = g(x)$ is a differentiable function whose range is an interval I, and f is continuous on I, then*

$$\int f(g(x)) \cdot g'(x)\,dx = \int f(u)\,du$$

Consider the integral in the form $\int f(g(x)) \cdot g'(x)\,dx$. First, identify what f and g are.

(1) Substitute $u = g(x)$ and $du = \frac{du}{dx}dx = g'(x)\,dx$ to obtain $\int f(u)\,du$.
(2) Integrate with respect to u.
(3) Replace u by $g(x)$.

Example. (Trigonometry)

(1) $\int 5\sec^2(5x+1)\,dx$.
(2) $\int \cos(7\theta+3)\,d\theta$.

Example. (Less obvious substitutions)

(1) $\int 3x^2 e^{x^3}\,dx$.
(2) $\int x\sqrt{2x+1}dx$.
(3)